Compositions (and Decompositions) of Functions

A "composition" can mean many things. It can mean a work of art, music, or writing. It can be a way of combining molecules. It can even be (apparently) a type of doll. In mathematics, it refers to a very specific way of combining functions. There are five main ways to combine two functions to make a new function:

(1) add them
(2) multiply them
(3) subtract them
(4) divide them
(5) *compose them*

The first four ways are "arithmetic" combinations of functions, which we've talked about. The last way, composing functions, is the most complicated, so it gets a section all to itself.

Remember how functions can be viewed as machines? They have inputs, they process the inputs somehow, and they spit out outputs. They're mathematical machines. Composing two functions is like linking two machines together. It's taking an input, running it through one machine, taking the *output* of that machine and running that through the second machine. In other words, the output of the first machine becomes the input of the second machine. In mathematical notation, we can write it this way:

Given two functions, f and g, the composition $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$.

Here, g is the first machine and f is the second. Let's decipher the notation just a bit. The notation $f \circ g$ is just how we write the composition of these two functions, and we say "f of g." On the other side of the equation we have $f(g(x))$. Here, you can think of x as an input into the function g. Then $g(x)$ is the corresponding output. Then we're plugging $g(x)$ *into* the function f; in other words, we're writing $g(x)$ as an input for the function f, so we get $f(g(x))$.

There are a few ways we can try to make this more concrete. Let's take two functions as an example. Let's say $f(x) = 3x - 2$ and $g(x) = x^2 + 1$. First, let's compute the composition of these two functions for a few specific inputs. What if we want to know "f of g" for the input 3? To compute "f of g," we first plug 3 into the function g, to get $g(3) = 3^2 + 1 = 10$. So $g(3) = 10$. We then take that output, and use it as an input to the function f. So we plug 10 into f, and get $f(10) = 3 \cdot 10 - 2 = 28$. Therefore, $(f \circ g)(3) = 28$. Let's do another one. What if we want to know $(f \circ g)(1)$? We first compute $g(1) = 2$, then plug that output into f, to get $f(2) = 4$, so in other words,

$$(f \circ g)(1) = f(g(1)) = f(2) = 4.$$

So computing a composition of functions for specific inputs is just a matter of doing some plugging in, or evaluating functions at certain points (since you might not always have a formula for each function). Try some.

(1) Let $f(x) = 2x^2$ and $g(x) = x + 3$. Find the following values.

(a) $(f \circ g)(-1)$ $= f(g(-1)) = 8$

(b) $(g \circ f)(-1)$ $= 5$

(c) $(g \circ g)(2)$ $= 8$

(2) Suppose $f(1) = 2$, $f(0) = 5$, $g(2) = 6$, g(3)=7 and $g(-3) = 0$. Find the following values.

(a) $(f \circ g)(-3)$. $= 5$

(b) $(g \circ f)(1)$. $= 6$

$f(x)=2x \quad g(x)=(x+1)^2$

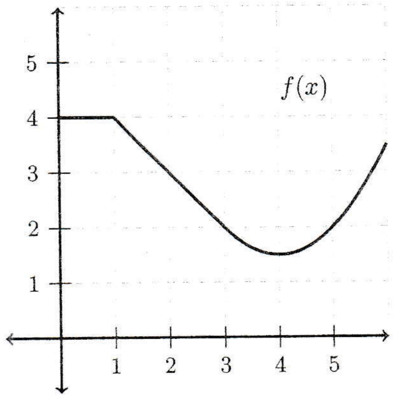
(3) Suppose f is the function that takes a number and doubles it and g is the function that adds 1 to a number and then squares that sum. Find the following values.

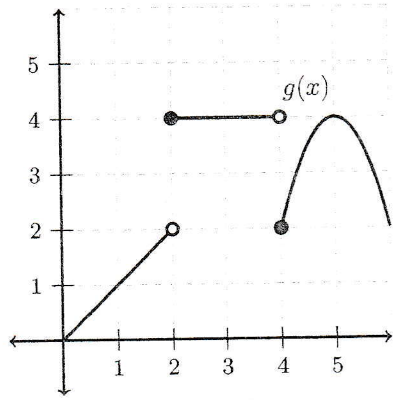
(a) $(f \circ g)(1)$ $= 8$

(b) $(g \circ f)(-2)$ $=9$

(c) $(f \circ f)(3)$ $=12$

(4) Let $f(x)$ and $g(x)$ be functions defined on [0,5] with the graphs shown below. Use the graphs to evaluate the following.





(a) $(f \circ g)(1)$ $=4$

(b) $(f \circ f)(2)$ $=2$

(c) $(g \circ f)(5)$ $=4$

Now, what if we want a *formula* for the composition of two functions? We have a formula for f ($f(x) = 3x - 2$), and we have a formula for g ($g(x) = x^2 + 1$), so conceivably, we should be able to find a formula for $f \circ g$. Turns out this isn't too hard, either. Imagine what we're doing – we're taking some arbitrary number (call it x), and plugging it into g. The output of g is going to be $x^2 + 1$. That output we're going to take and plug into the function f. What f does is take an input, multiply it by 3, and then subtract 2 (because the formula for f is $f(x) = 3x - 2$). So if we plug in $x^2 + 1$ to f, we get $3(x^2 + 1) - 2$. Therefore,

$$(f \circ g)(x) = 3(x^2 + 1) - 2 = 3x^2 + 1$$

Notice that we could compute $(f \circ g)(3)$ or $(f \circ g)(1)$ using this formula instead of the step-by-step process we did above, and we'll get the same answers:

$$(f \circ g)(3) = 3(3^2) + 1 = 3 \cdot 9 + 1 = 28$$
$$(f \circ g)(1) = 3(1^2) + 1 = 3 \cdot 1 + 1 = 4$$

So now we know what a composition of two functions means, we know how to compute it for specific inputs, and we know how to compute a formula for it if we have formulas for each of the individual functions. Now for a few facts about compositions of functions.

- **The Order Matters.** Composition of functions is doing things one at a time. When you do things one a time, the order you do them matters. If you put on your socks first and then your shoes, you will have a very different result than if you do things the other way around.
- **Things Can Get Tricky with Domains** Try this example. Let $f(x) = \frac{1}{x}$, and let $g(x) = 1 - x$. The number 1 is in the domain of f, and it's in the domain of g. Is it in the domain of $f \circ g$? What about $g \circ f$?

$(f \circ g)(1) = f(0) = \frac{1}{0}$ = undefined, so 1 is not in domain of $f \circ g$

$(g \circ f)(1) = g(1) = 1 - 1 = 0$ ✓, so 1 is in domain of $g \circ f$

Here's some practice finding formulas for compositions and determining what falls in their domains.

(5) Let $f(x) = \sqrt{x-5}$ and $g(x) = x^2+1$. Find the formulas for the following functions.

(a) $f \circ g = \sqrt{x^2+1-5} = \boxed{\sqrt{x^2-4} = (f\circ g)(x)}$

(b) $g \circ f = (\sqrt{x-5})^2+1 = x-5+1 = \boxed{x-4 = (g\circ f)(x)}$

(c) $g \circ g = (x^2+1)^2+1 = (x^2+1)(x^2+1)+1 = \boxed{x^4+2x^2+2 = (g\circ g)(x)}$

(6) Let $f(x) = \dfrac{1}{x+1}$ and $g(x) = \dfrac{1}{x-1}$.

(a) What is $f \circ g$? $\dfrac{1}{\frac{1}{x-1}+1} = \dfrac{1}{\frac{1}{x-1}+\frac{x-1}{x-1}} = \dfrac{1}{\left(\frac{x}{x-1}\right)} = \dfrac{x-1}{x}$

(b) Is -1 in the domain of $f \circ g$?

$f(g(-1)) = f(-\frac{1}{2}) = 2$ ✓ yes -1 is in domain of $f\circ g$

(c) Is 0 in the domain of $f \circ g$?

$f(g(0)) = f(-1) = \frac{1}{0}$ = undefined, no 0 is not in domain of $f\circ g$

Perhaps even more often than composing functions, we want to decompose functions. That is, we want to take a complicated function and break it down into smaller parts by writing it as the composition of two simpler functions. For example, maybe we have the function:

$$h(x) = \sqrt{3x-7}$$

That might be kind of complicated to deal with, and many times (most especially in calculus) we want to decompose it into smaller parts. Like breaking down a molecule to look at its atoms. So we could decompose h as $f \circ g$, where $f(x) = \sqrt{x}$ and $g(x) = 3x-7$. Knowing how to decompose functions properly is a very important skill to learn for calculus.

Try decomposing the following functions. Find two functions, f and g, such that $h(x) = (f \circ g)(x)$. Just to mess with you, we'll throw in one that doesn't actually decompose in this way.

(7) $h(x) = 5x+6$

$f(x) = x+6 \quad g(x) = 5x$

(8) $h(x) = \dfrac{1}{x^2+1}$

$f(x) = \dfrac{1}{x} \quad g(x) = x^2+1$

(9) $h(x) = x + \frac{1}{x}$

Does not decompose

(10) $h(x) = \frac{1}{x} - 1$

$f(x) = x-1 \quad g(x) = \dfrac{1}{x}$

(11) Find *three* functions whose composition is $h(x) = \dfrac{2}{\sqrt{x^2+1}-3}$. Can you find four? Five?

$f(x) = \dfrac{2}{x-3} \quad g(x) = \sqrt{x^2+1}$

$f(x) = \dfrac{2}{\sqrt{x+1}-3} \quad g(x) = x^2$

$f(x) = \dfrac{2}{x} \quad g(x) = \sqrt{x^2+1}-3$